The Ulam spiral functions (mod 210).

Overview: the Ulam spiral.

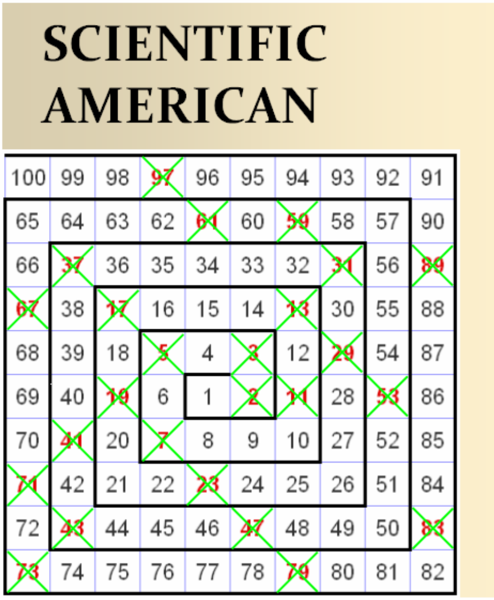

Fig. 1. the Ulam spiral on the cover of Scientific American (March 1964).

The Ulam spiral (1963) is a graphical depiction of the distribution of prime numbers. The spiral emphasizes the appearance of prominent diagonal, horizontal, and vertical lines containing large numbers of primes.
The patterns in the Ulam spiral are one of the great unsolved mysteries in mathematics.

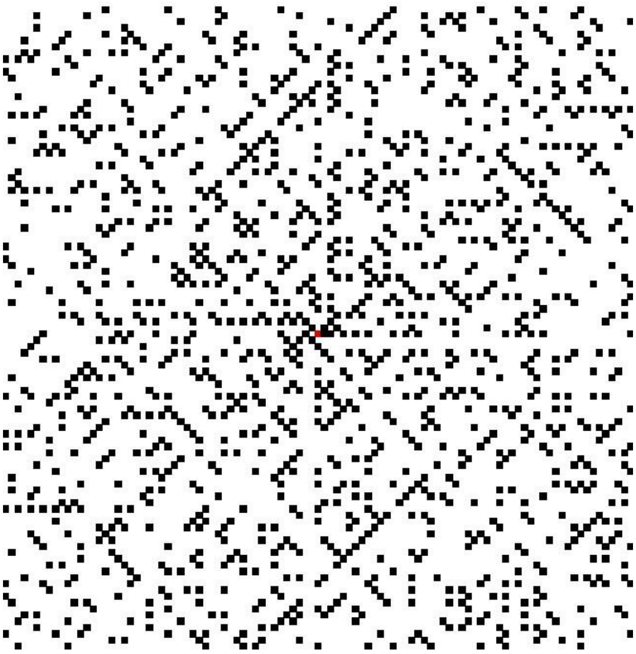

Fig. 2: a 101 x 101 counterclockwise Ulam spiral

The counterclockwise Ulam spiral with startvalue 0 can be build using eight families of functions that capture all natural numbers. The functions are $f_{b,c}(n) = 4n^2 + bn + c$, with $n \in \mathbf{N}_0$, $b, c \in \mathbf{Z}$ and $-3 \le b \le 4$.
For instance, the SW diagonals appear onwards from the intersection with the line $y = -x$, and are member of the $f_c(n_{SW}) = 4n^2 + 2n + c$ family of functions. Fig. 2 clearly shows the high density of prime numbers on the diagonal belonging to the $f_{41}(n_{SW}) = 4n^2 + 2n + 41$, and also of the $f_{41}(n_{NE}) = 4n^2 - 2n + 41$ function.

Modulo 210 calculations.

To shed some light on the conundrum of the diagonals, the families of functions $f_{b,c}(n) = 4n^2 + bn + c$ are combined with the P_4#−sieve. The P_4#−sieve has $\varphi(p_4\#) = 48$ struts relative prime to the primorial p_4#, the product of the first four prime numbers. The function $f_{b,c}(n)$ (mod 210) has a residue class of just 24 elements.
Diagonals may have a high ratio of prime numbers when the residue class solely alines with struts.
Fig. 3a shows the distribution of the residue class of the function $f_{41}(n_{SW})$ (mod 210), with $f_{41}(n_{SW}) < 10^9$.
Fig. 3b shows a similar distribution for the function $f_{59}(n_{SE})$ (mod 210).

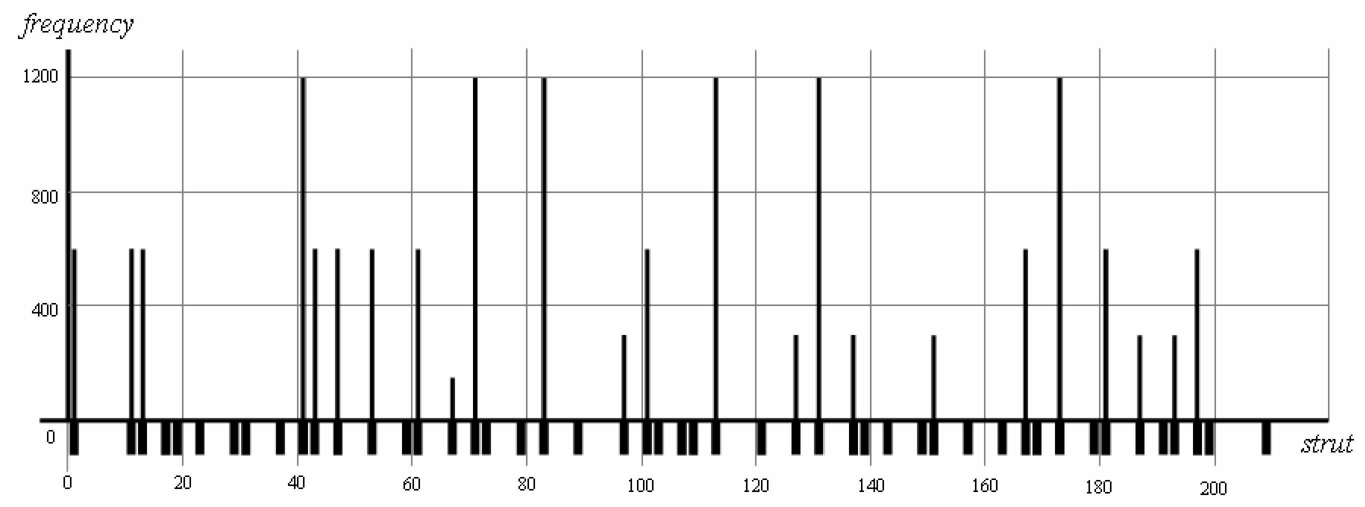

Fig. 3a: $f_{41}(n_{SW})$ (mod 210): position of the residue class of above the 48 struts of the P_4#−sieve.

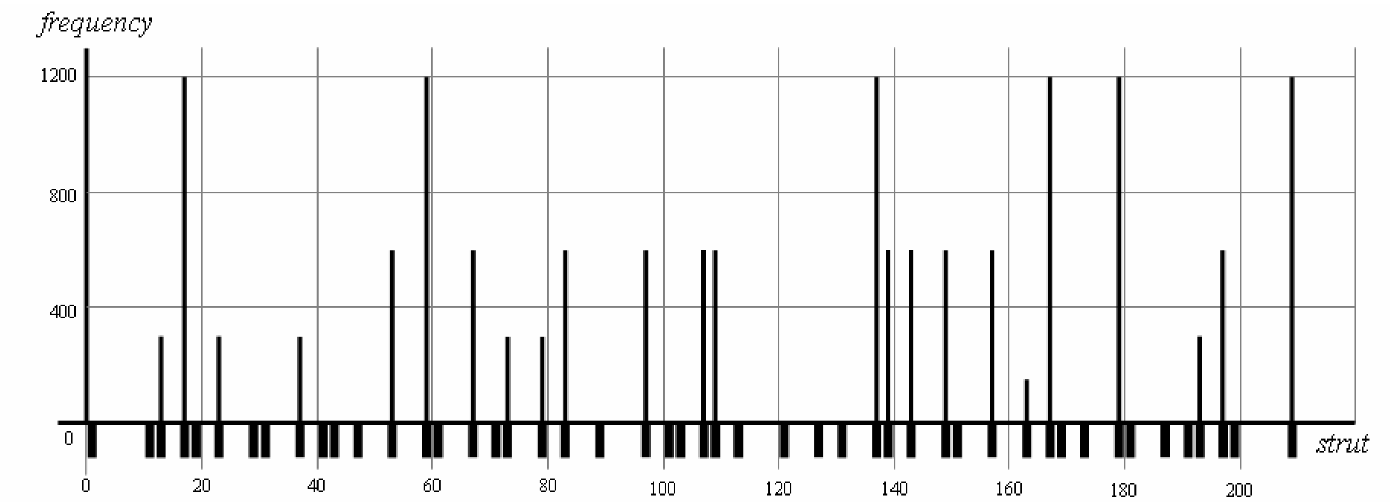

Fig. 3b: $f_{59}(n_{SE})$ (mod 210): position of the residue class above the 48 struts of the P_4#−sieve.

Implementing the residue class.

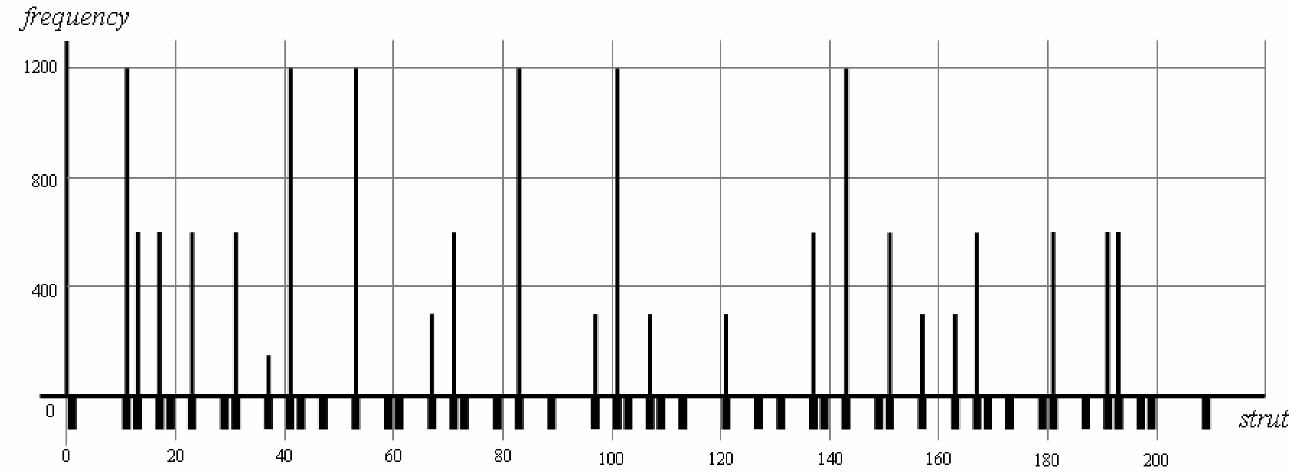

Fig. 4: $f_{37}(n_{NW})$ (mod 210): position of the residue class above the 48 struts of the P_4#−sieve.

The 24 members of the residue class of the function $f_{b,c}(n)$ (mod 210) have one distinct pattern. This unique pattern rotates over the base of the P_4#−sieve.
Fig. 4 shows the distribution of the residue class of the function $f_{37}(n_{NW})$ (mod 210), with $f_{37}(n_{NW}) < 10^9$.
The same unique pattern is recognizable in fig. 3a and fig. 3b.

The function $f_c(n_{NW}) = 4n^2 + 0n + c$ has 6 values of c (mod 210) whereby all 24 members in the residue class aline with struts of the P_4#−sieve, see the table below.
It applies that: $R(f_c(n_{NW})) \equiv R(f_{c+104}(n_{SE})) \equiv R(f_{c+26}(n_{NE})) \equiv R(f_{c+26}(n_{SW}))$ (mod 210), with R the residue class.
The distribution of the residue class of the function $f_{41}(n_{SW})$ (mod 210) in fig. 3a thus also belongs to the function $f_{67}(n_{NW})$ (mod 210).

The frequency = 151 of $f_{37}(n_{NW})$ (mod 210) at strut = 37 (see fig. 4) shifts to strut = 67 for $f_{67}(n_{NW})$ (mod 210) (see fig. 3a), etc.

$R(f_{b,c}(n)) \equiv$ 24 Struts (mod 210)	b	c	c	c	c	c	c
$f_c(n_{NW})$	0	37	43	67	127	163	193
$f_c(n_{NE})$	−2	11	17	41	101	137	167
$f_c(n_{SW})$	2	11	17	41	101	137	167
$f_c(n_{SE})$	4	143	149	173	23	59	89

High ratios of prime numbers in $f_{b,c}(n) = 4n^2 + bn + c$ may be found when the residue class (mod 210) fully alines with struts of the P_4#−sieve, see the table below.

Function, with $f_{b,c}(n) < 10^9$	$f_{41}(n_{SW})$	$f_{251}(n_{SW})$	$f_{-169}(n_{SW})$	$f_{-457}(n_{SE})$	$f_{487}(n_{NW})$
Prime number ratio	0.361	0.215	0.289	0.256	0.266

Function, with $f_{b,c}(n) < 10^9$	$f_{59}(n_{SE})$	$f_{-361}(n_{SE})$	$f_{-151}(n_{SE})$	$f_{-283}(n_{SW})$	$f_{163}(n_{NW})$
Prime number ratio	0.321	0.348	0.184	0.318	0.359

Function, with $f_{b,c}(n) < 10^9$	$f_{23}(n_{SE})$	$f_{-397}(n_{SE})$	$f_{653}(n_{SE})$	$f_{-109}(n_{SW})$	$f_{-293}(n_{NW})$
Prime number ratio	0.193	0.395	0.362	0.300	0.264

Function, with $f_{b,c}(n) < 10^9$	$f_{143}(n_{SE})$	$f_{-907}(n_{SE})$	$f_{-6367}(n_{SE})$	$f_{-619}(n_{SW})$	$f_{37}(n_{NW})$
Prime number ratio	0.245	0.326	0.397	0.339	0.256