Properties of the segmented prime spirals.

1. Abstract.

The segmented prime spirals is a way to visualize the distribution of prime numbers amongst a sequential set of natural numbers. The segmented prime spiral consists of segments of sequential natural numbers, who together with the other segments form a continuous spiral of natural numbers. There are infinitely many segmented prime spirals.
In the segmented prime spirals the prime numbers have the tendency to line up along specific odd diagonals, while other odd diagonals hardly contain any prime numbers.
The Ulam spiral, as discovered by Stanislaw Ulam in 1963, is a special seqential prime spiral and has four segments.

2. Mathematical properties of the segmented prime spiral.

The counterclockwise prime spiral with startvalue 0 and m segments is fully defined by the $(2m + 1)$ families of quadratic functions $f_{a,b,c}(n) = an^2 + bn + c$, with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$, $a = m$, $-a \le b \le a$ with $b \in Z$, and

$$\begin{cases} c \in Z_0^- & \text{if } b = a \\ c \in Z & \text{if } -a < b < a \\ c \in Z^+ & \text{if } b = -a \end{cases}$$

For $b = a$ the function $f_{a,b,c}(n) = an^2 + bn + c$ becomes $f_{a,b,c}(n) = an^2 + an + c$.
The translation $n \longmapsto n - 1$ then gives the function $f_{a,b,c}(n) = an^2 - an + c$ and thus $f_{a,b,c}(n) = an^2 - bn + c$.
The functions $f_{a,b,c}(n) = an^2 + bn + c$ and $f_{a,b,c}(n) = an^2 - bn + c$ give equal results, but for the translation $n \longmapsto n - 1$.

Fig 2.1 shows an example of a counterclockwise prime spiral with five segments. The startvalue 0 is in the center of the spiral, with the value 1 a single step due east.
The lines belonging to the functions $f_{5,b,0}(n) = 5n^2 + bn + 0$ contain no prime numbers $> p_4$.
The diagonal $f_{5,-5,1}(n) = 5n^2 - 5n + 1$ with the sequence $\{1, 11, 31, 61, ...\}$ appears to be rich with prime numbers.

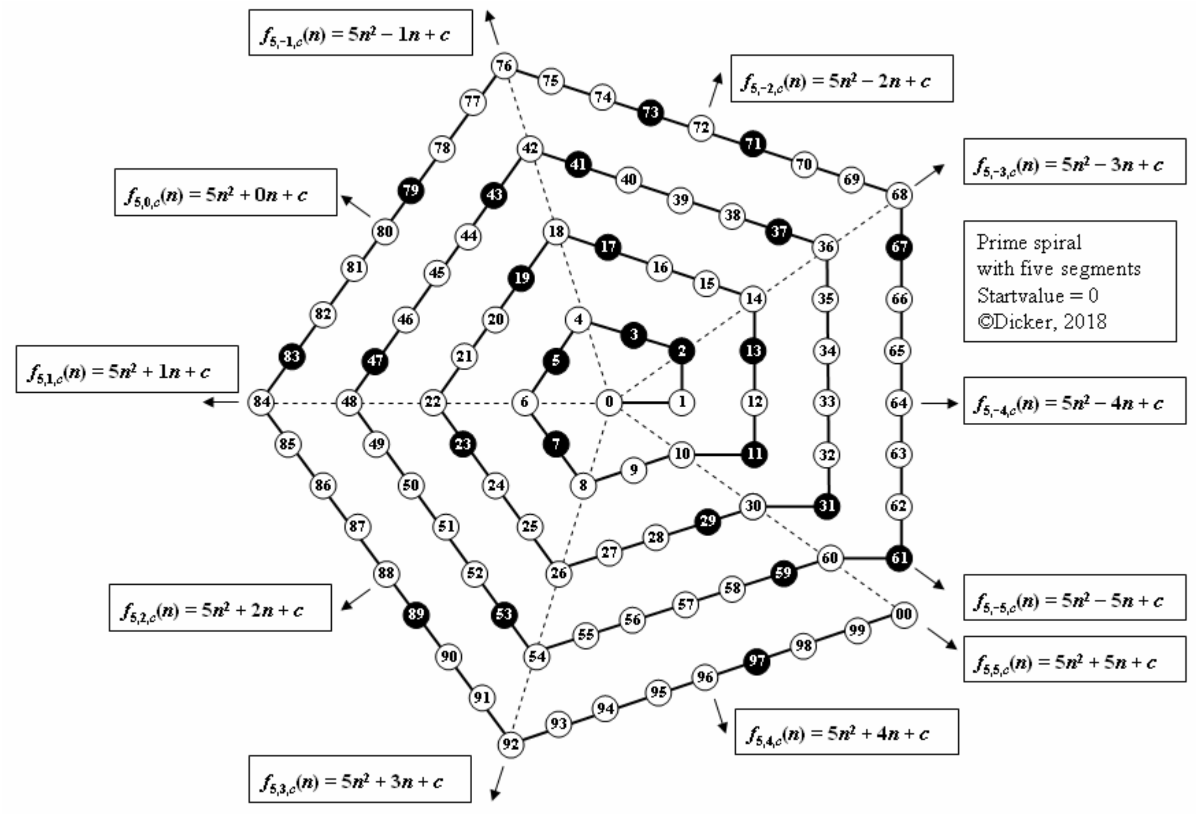

Fig. 2.1: A counterclockwise prime spiral with startvalue 0 and five segments.

3. The Ulam spiral as prime spiral with four segments.

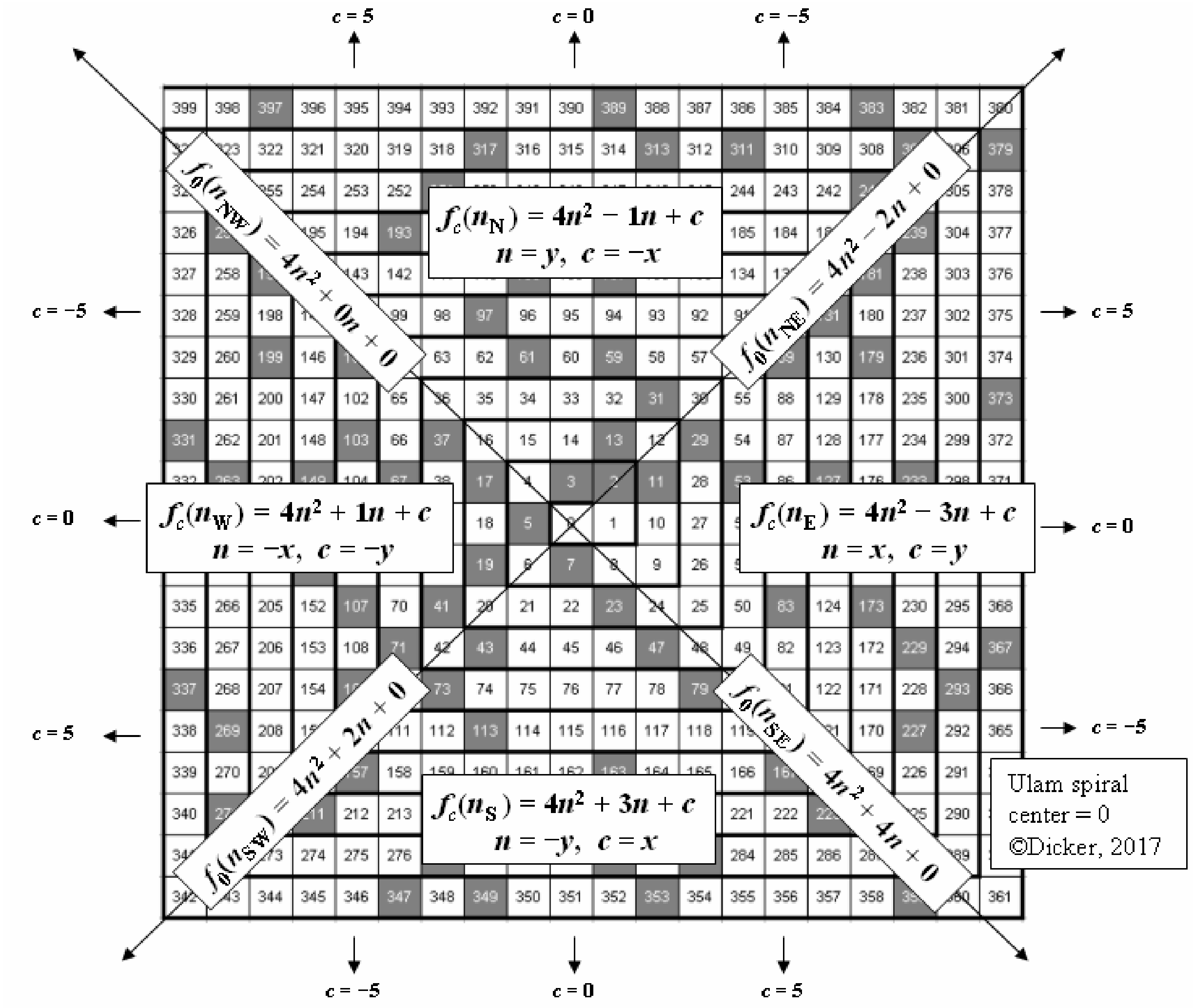

Fig. 3.1: The Ulam spiral and the eight families of functions.

Fig. 3.1 shows the counterclockwise Ulam spiral with startvalue 0 when placed in a Cartesian coordinate system. All natural numbers in the spiral are completely captured by eight families of quadratic functions. The SE diagonal is defined by the family of functions $f_c(n_{SE}) = 4n^2 + 4n + c$, or the more general function $f_{4,4,c}(n) = 4n^2 + 4n + c$,

The counterclockwise Ulam spiral with startvalue 0 is a four-quarter Ulam spiral, and thus a prime spiral with four segments. Per definition of the segmented prime spiral the Ulam spiral is defined by nine families of functions. Clearly visible is the discrepancy when counterclockwise crossing the SE main diagonal. For positive values of c the SE diagonal becomes the function $f_{4,-4,c}(n) = 4n^2 - 4n + c$. Using the function $f_c(n_{SE}) = 4n^2 + 4n + c$ instead for all SE diagonals gives almost identical results.

4. Prime spirals with three segments.

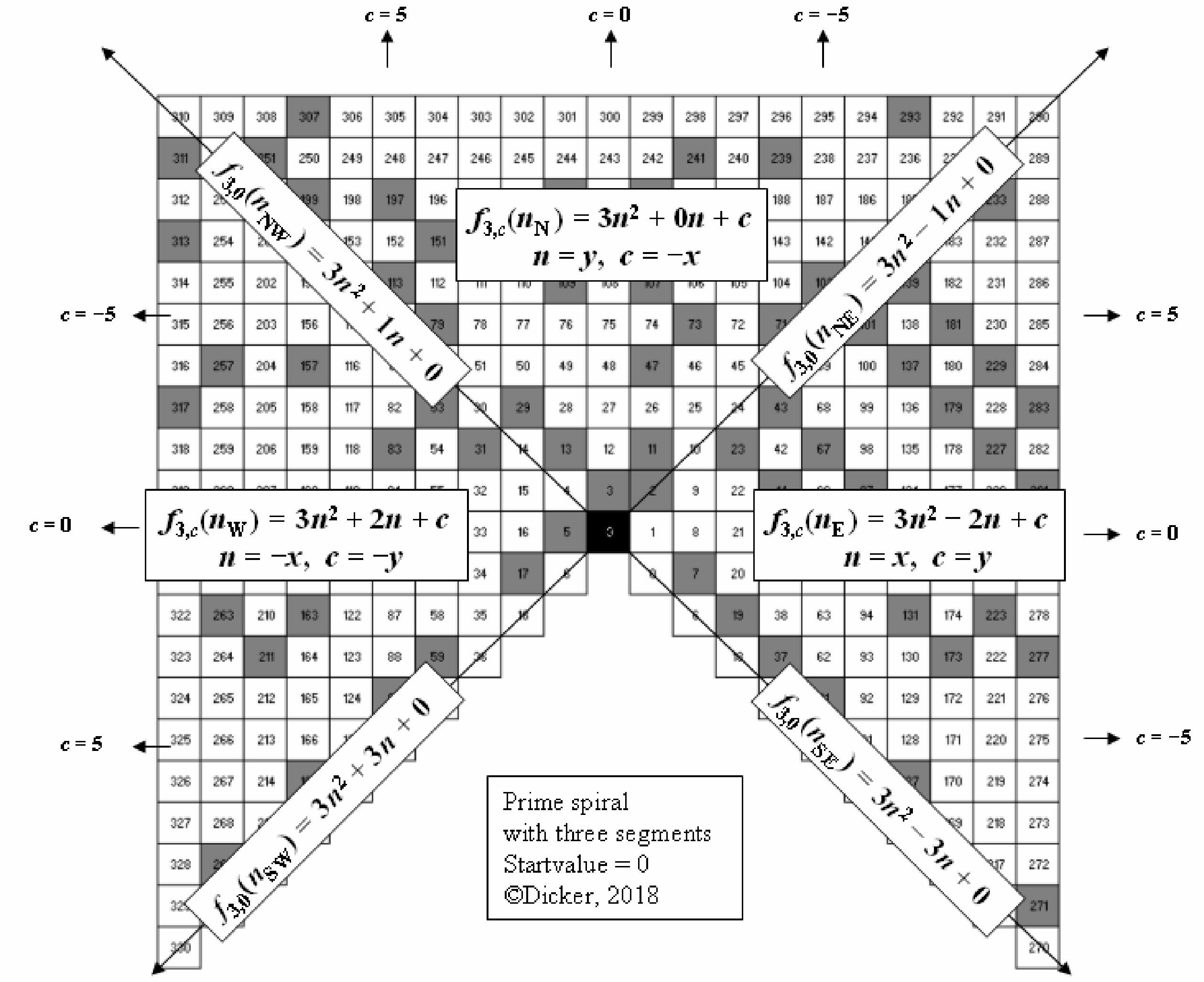

Fig. 4.1: A prime spiral with three segments based on the Ulam spiral.

The segmented prime spirals are an offspring of the Ulam spiral.
The counterclockwise Ulam spiral with startvalue 0 is a four-quarter spiral, and thus a prime spiral with four segements. Downwards is the prime spiral with three segments, which can be visualized as a three-quarter Ulam spiral (fig. 4.1).

For the counterclockwise prime spiral with three segments, the general families of functions $f_{a,b,c}(n) = an^2 + bn + c$ thus become $f_{3,b,c}(n) = 3n^2 + bn + c$. When using the compass rose, the value of b is synonymous to the winddirection.
Almost each function $f_{a,b,c}(n) = an^2 + bn + c$ appears as a horizontal, vertical or diagonal line only at a higher n.
For instance the sequence {43, 67, 97, 133, 175, 223, 277, ...} of the function $f_{3,7}(n_{SE}) = 3n^2 - 3n + 7$ (also known as $f_{3,-3,7}(n) = 3n^2 - 3n + 7$) starts at $n = 4$ and thus as part of the function $f(n) = 3n^2 + 21n + 43$.

The function $f_{3,c}(n_{SW}) = 3n^2 + 3n + c$ with $c \in Z_0^-$ is separted from the function $f_{3,c}(n_{SE}) = 3n^2 - 3n + c$ with $c \in Z^+$.
Within the families of functions $f_{3,c}(n_{SE}) = 3n^2 + 3n + c$ and $f_{3,c}(n_{SW}) = 3n^2 - 3n + c$, every third odd diagonal, like {15, 33, 57, 87, ...} and {9, 21, 37, 63, ...}, contains no prime numbers. For $3 \mid c$ applies $3 \mid f_{3,c}(n_{SE}) = 3n^2 - 3n + c$.

4.1. Coordinates of a natural number in the counterclockwise three-quarter Ulam spiral.

When the counterclockwise prime spiral with three segments is presented as a three-quarter Ulam spiral in a Cartesian coordinate system, it is possible to calculate the (x, y) coordinates of any natural number g.
The technique is the same as applied in the counterclockwise Ulam spiral with startvalue 0.

Define $m = \sqrt{g\ /\ 3}$ with $m \in \mathbf{R}_{\geq 0}$ and $n = \lfloor\, m\, \rceil$ with $n \in \mathbf{N_0}$.
The value $m - n$ determines the sector in which g lies, see the table below.

$m - n$	Sector	Function	(x, y)
$-^3/_6 \leq m - n < -^1/_6$	E	$f_{3,c}(n_E) = 3n^2 - 2n + c$	(n, c)
$-^1/_6 \leq m - n < ^1/_6$	N	$f_{3,c}(n_N) = 3n^2 + 0n + c$	$(-c, n)$
$^1/_6 \leq m - n < ^3/_6$	W	$f_{3,c}(n_W) = 3n^2 + 2n + c$	$(-c, n)$

In the next table are the positions of some natural numbers from the segmented prime spiral of fig. 3.
Calculations specifically make the natural number 270 a member of the function $f_{3,0}(n_W) = 3n^2 + 2n + 0$ and not the function $f_{3,0}(n_E) = 3n^2 - 2n + 0$, which complies with the definition of the families of functions.

g	m	n	Function	c	(x, y)
270	9.486…	9	$f_{3,c}(n_W) = 3n^2 + 2n + c$	9	$(-9, 9)$
271	9.504…	10	$f_{3,c}(n_E) = 3n^2 - 2n + c$	−9	$(10, -9)$
290	9.831…	10	$f_{3,c}(n_E) = 3n^2 - 2n + c$	10	$(10, 10)$
291	9.848…	10	$f_{3,c}(n_N) = 3n^2 + 0n + c$	−9	$(9, 10)$
311	10.181…	10	$f_{3,c}(n_W) = 3n^2 + 2n + c$	−9	$(-10, 9)$
330	10.488…	10	$f_{3,c}(n_W) = 3n^2 + 2n + c$	10	$(-10, -10)$

4.2. Other graphical representations of the prime spiral with three segments.

The Ulam three-quarter spiral (fig. 4.1) becomes a continuous spiral (see fig. 4.2) when the prime spiral is folded together. Clearly visible is the translation $n \mapsto n-1$ at the original seam.

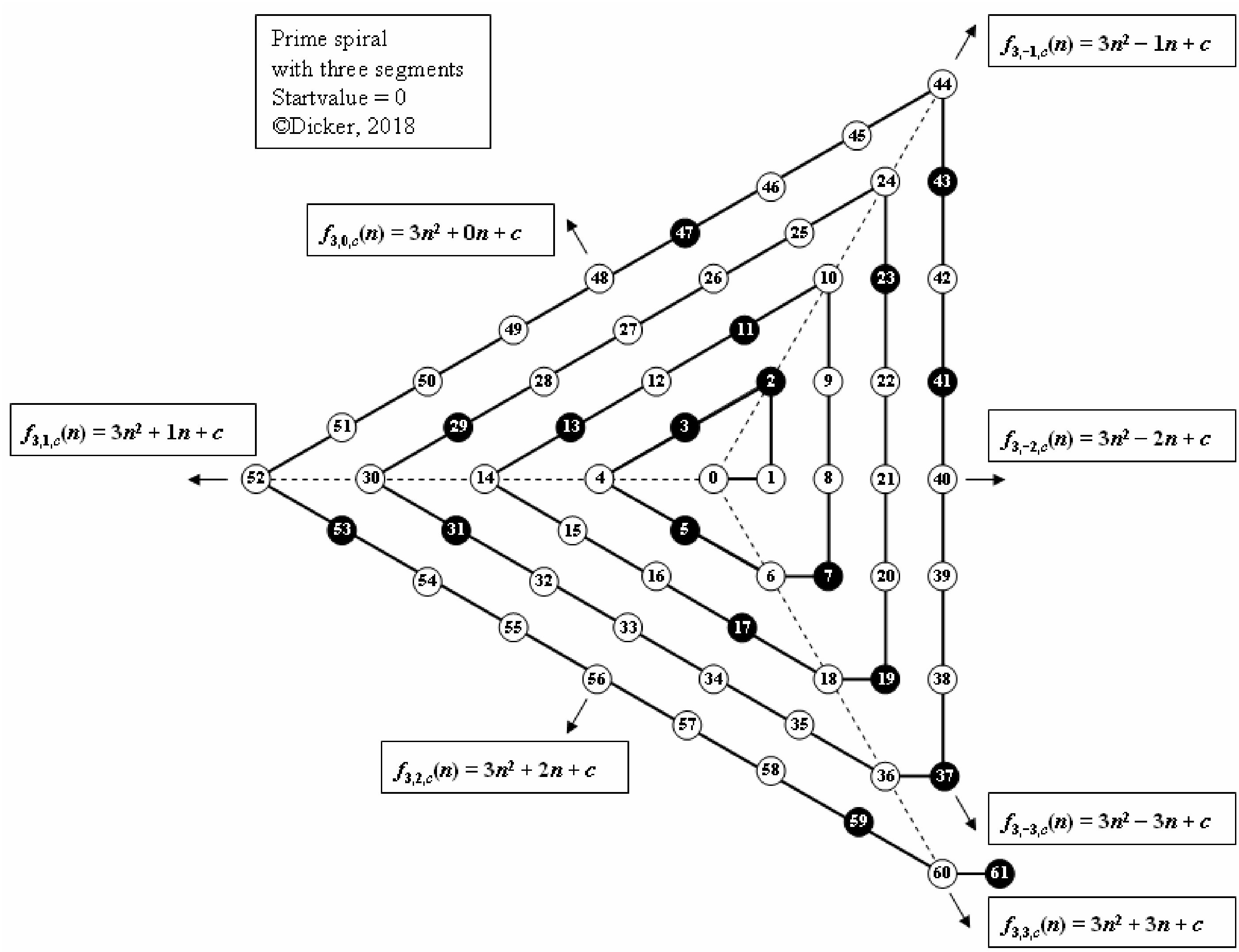

Fig. 4.2: A continuous prime spiral with three segments.

The Ulam three-quarter spiral (fig. 4.1) becomes a continuous spiral (see fig. 4.3) when the prime spiral is presented as a hexagram. There is now a seemingly seamless transition from the function $f_{3,c}(n_{\mathrm{SW}}) = 3n^2 + 3n + c$ with $c \in Z_0^-$ into the function $f_{3,c}(n_{\mathrm{SE}}) = 3n^2 - 3n + c$ with $c \in Z^+$.

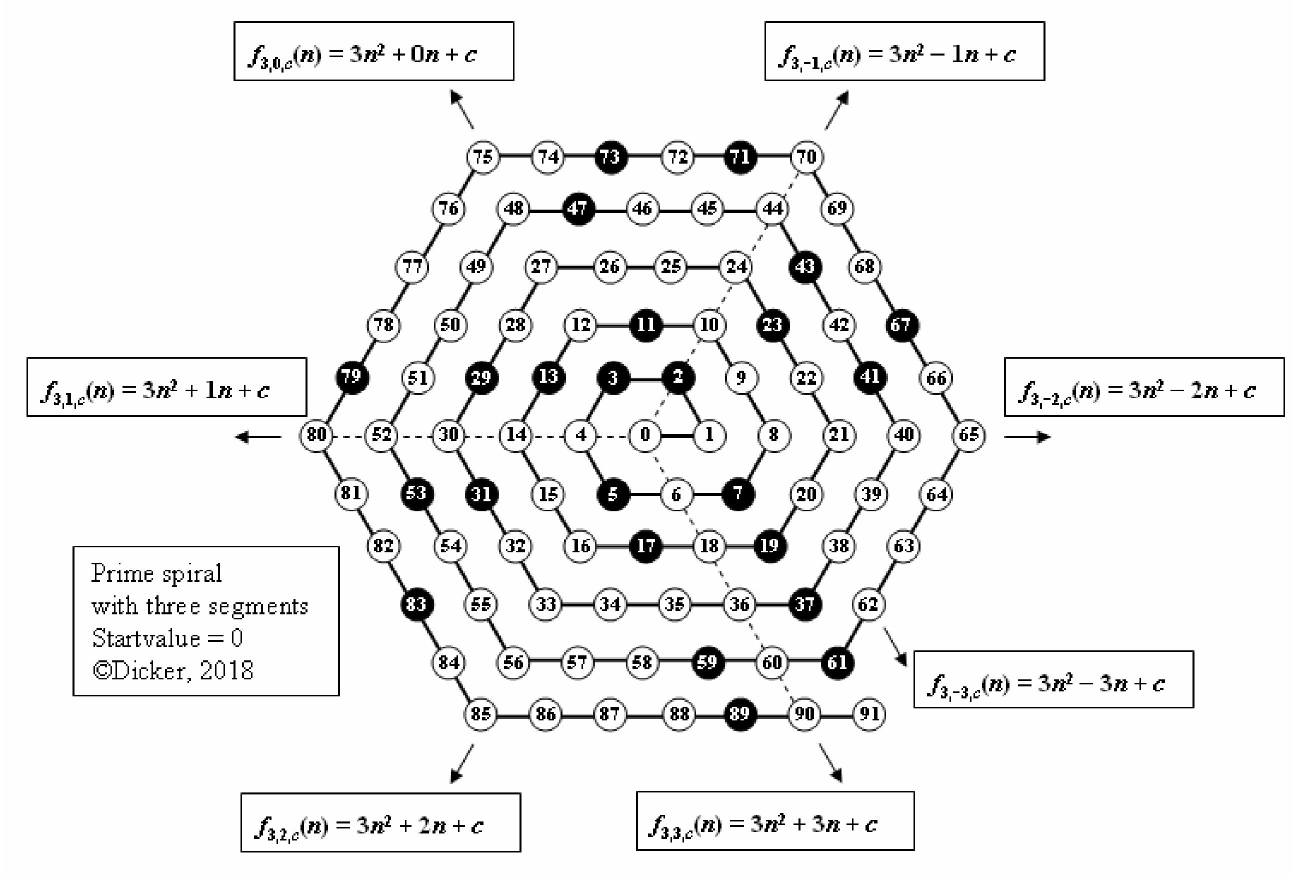

Fig. 4.3: A hexagram representation of a continuous prime spiral with three segments.

5. Prime spirals with two segments.

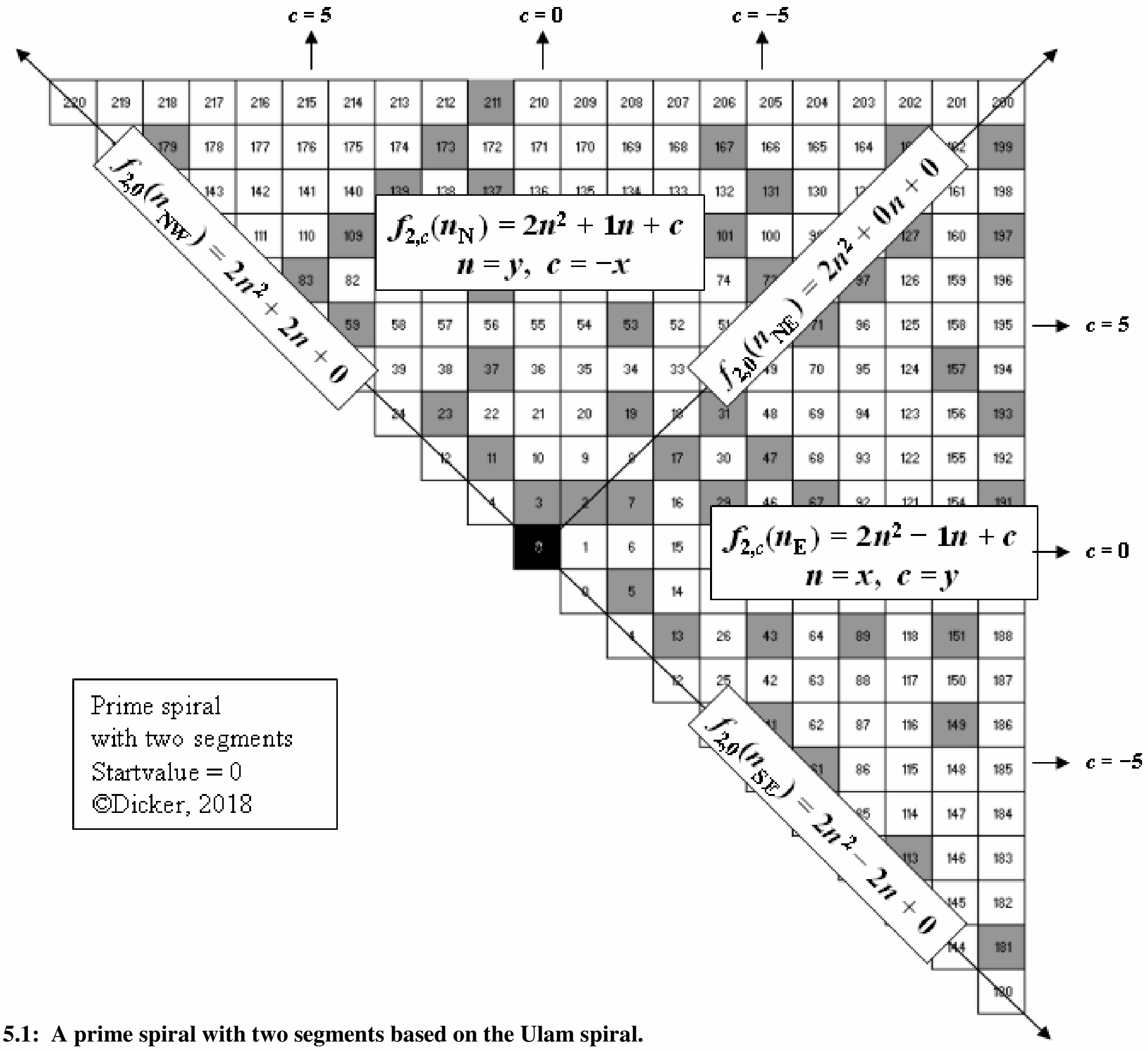

Fig. 5.1: A prime spiral with two segments based on the Ulam spiral.

In fig. 5.1 the prime spiral with two segments is visualized as a two-quarter Ulam spiral. For the counterclockwise prime spiral with startvalue 0 the general families of functions $f_{a,b,c}(n) = an^2 + bn + c$ thus become $f_{2,b,c}(n) = 2n^2 + bn + c$. When using the compass rose the value of b is synonymous to the winddirection.
Almost each function $f_{a,b,c}(n) = an^2 + bn + c$ appears as a horizontal, vertical or diagonal line only at a higher n.
For instance the sequence {61, 83, 109, 139, 173, 211, ...} of the function $f_{2,11}(n_{NE}) = 2n^2 + 0n + 11$ (also known as $f_{2,0,11}(n) = 2n^2 + 0n + 11$) starts at $n = 5$ and thus as part of the function $f(n) = 2n^2 + 20n + 61$. The function appears to be rich with prime numbers.

The function $f_{2,c}(n_{NW}) = 2n^2 + 2n + c$ with $c \in Z_0^-$ is separted from the function $f_{2,c}(n_{SE}) = 2n^2 - 2n + c$ with $c \in Z^+$.

For any natural number g in the prime spiral with two segments the coordinates in the Cartesian coordinate system can be calculated through the families of functions. Define $m = \sqrt{g / 2}$ with $m \in \mathbf{R}_{\geq 0}$ and $n = \lfloor m \rceil$ with $n \in \mathbf{N_0}$.
The value $m - n$ determines the sector in which g lies, see the table below.

$m - n$	Sector	Function	(x, y)
$-½ \leq m - n < 0$	E	$f_{2,c}(n_E) = 2n^2 - 1n + c$	(n, c)
$0 \leq m - n < ½$	N	$f_{2,c}(n_N) = 2n^2 + 1n + c$	($-c$, n)

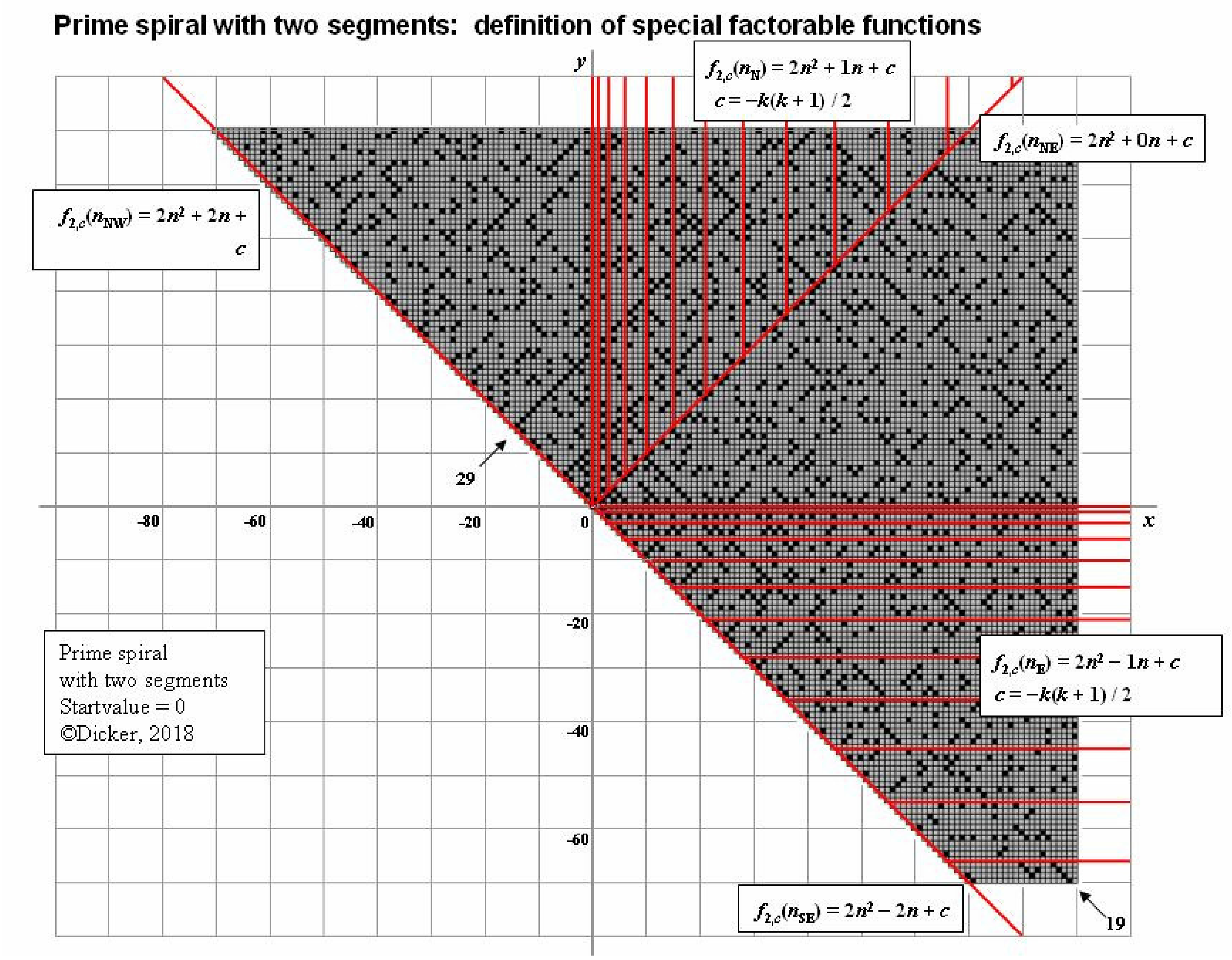

Fig. 5.2: Special factorable functions in the prime spiral with two segments.

Fig. 5.2 shows the special factorable functions that contain no prime numbers > p_2.
The function $f_{2,-1}(n_N) = 2n^2 + 1n - 1$ for instance, can be written as $f_{2,-1}(n_N) = (2n - 1)(n + 1)$.
When prime rich functions like $f_{2,29}(n_{NE}) = 2n^2 + 0n + 29$ or $f_{2,19}(n_{SE}) = 2n^2 - 1n + 19$ intersect with special factorable functions, natural numbers on the intersections are composite numbers.

5.1. Other graphical representations of the prime spiral with two segments.

The Ulam two-quarter spiral (fig. 5.1) becomes a continuous spiral (see fig. 5.3) when the prime spiral is folded together. Clearly visible is the translation $n \mapsto n-1$ at the original seam.

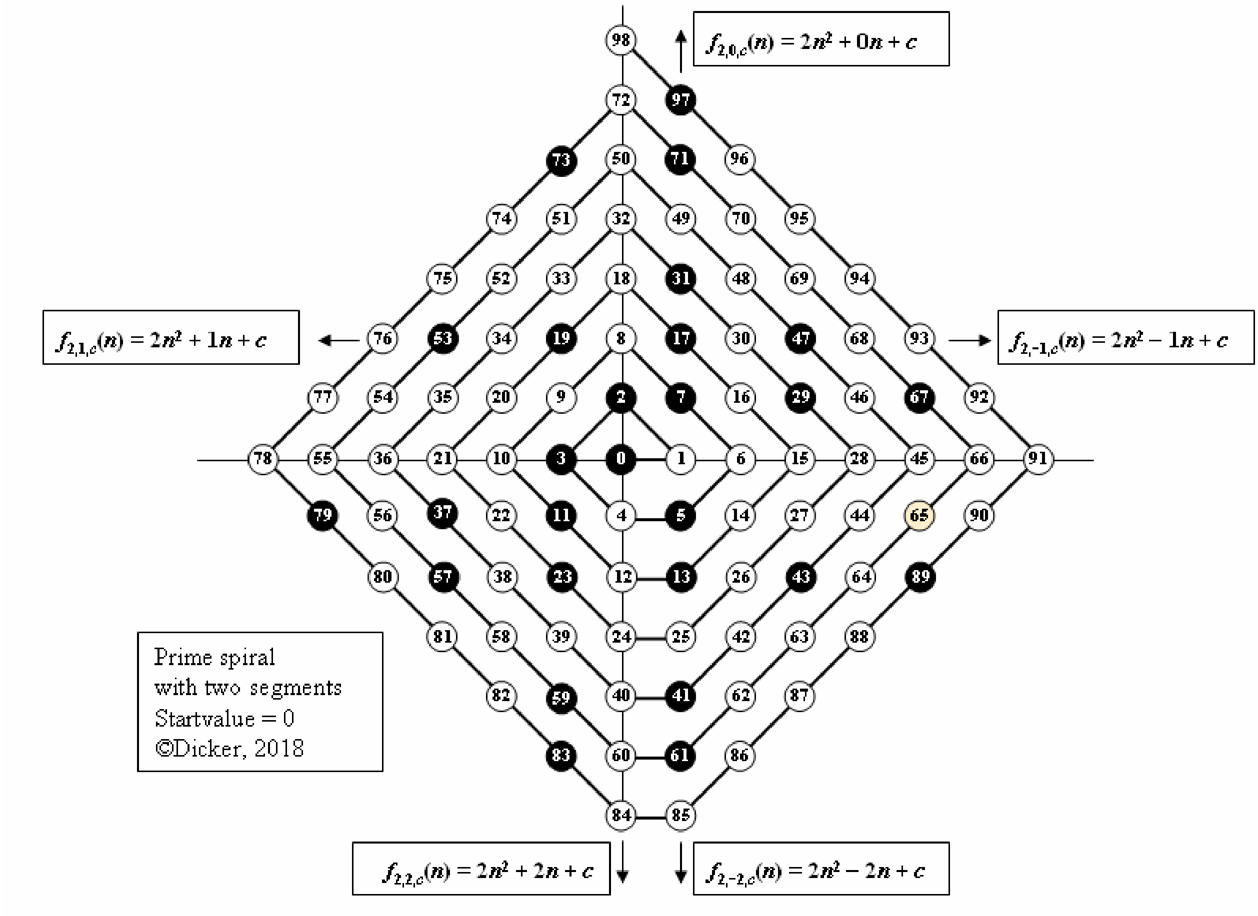

Fig. 5.3: A contiuous prime spiral with two segments.

The Ulam two-quarter spiral (fig. 5.1) becomes a continuous spiral (see fig. 5.4) when the prime spiral is presented in a diamond shape. Still visible is the translation $n \mapsto n-1$ at the original seam.

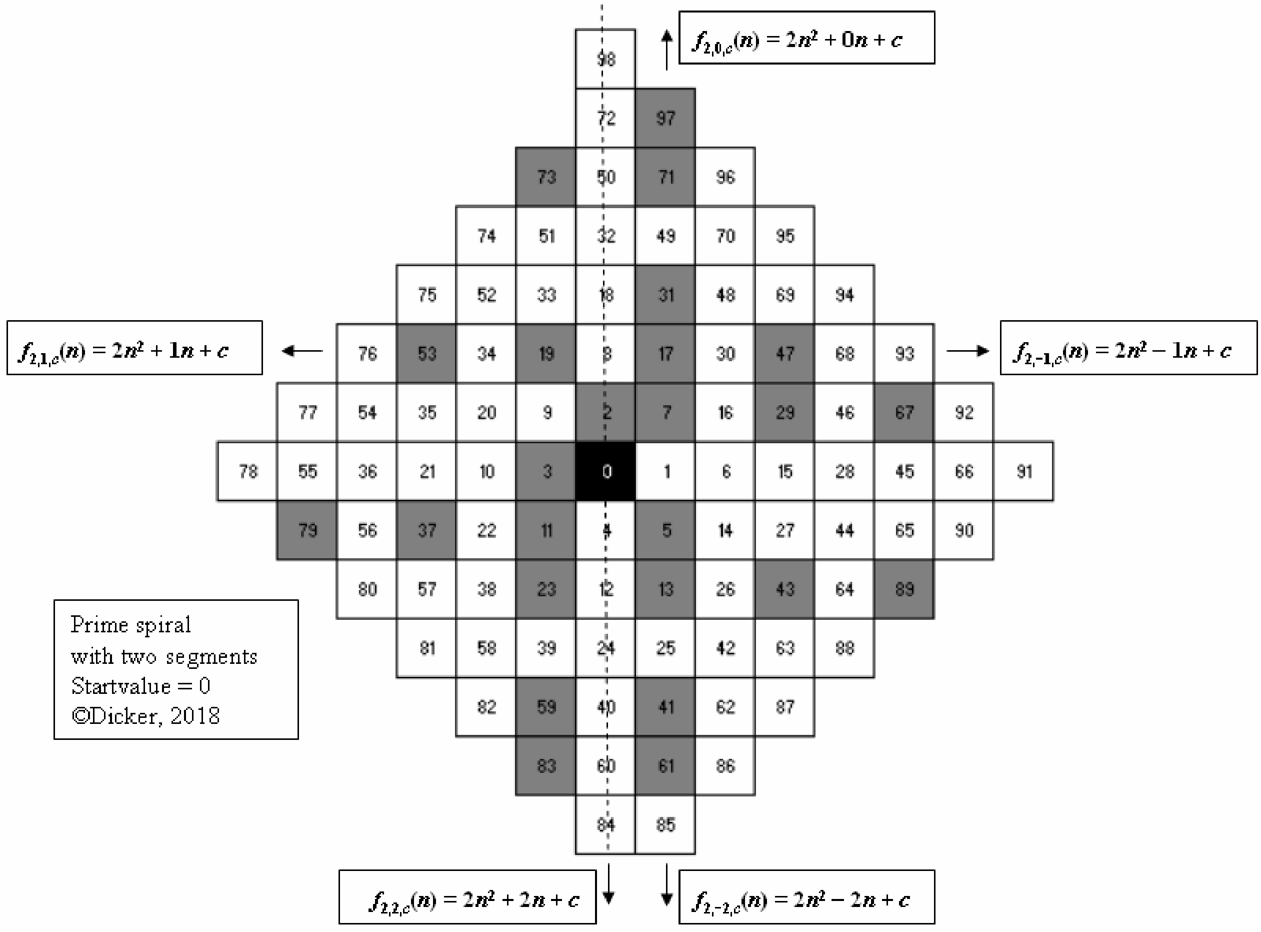

Fig. 5.4: An other representation of a continuous prime spiral with two segments.

6. Prime spirals with one segments.

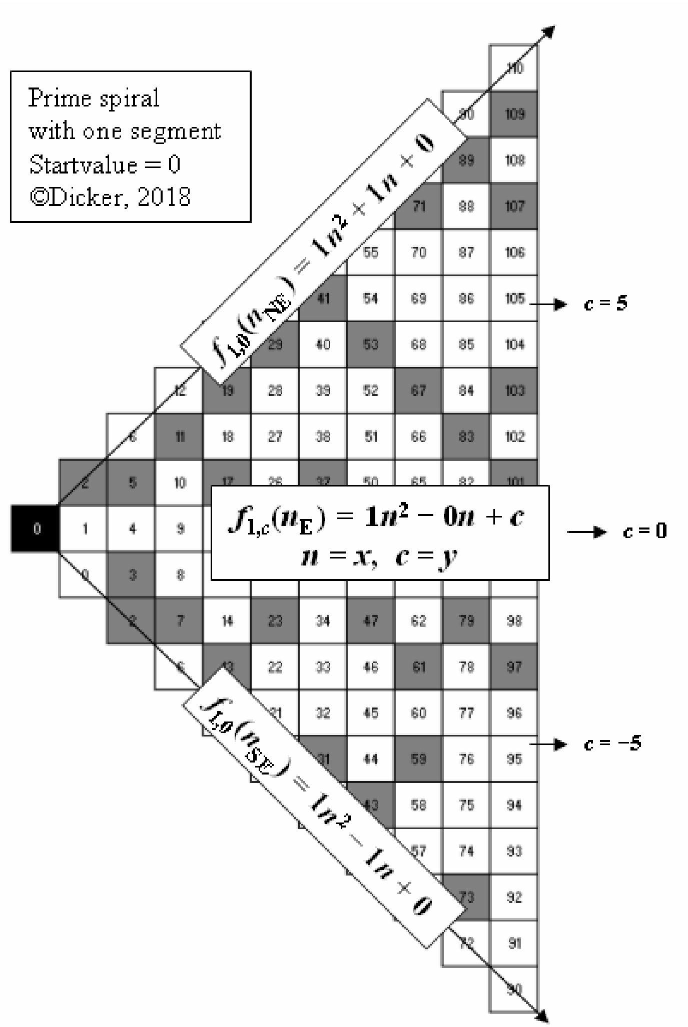

Fig. 6.1: A prime spiral with one segment based on the Ulam spiral.

In fig. 6.1 the prime spiral with one segment is visualized as a one-quarter Ulam spiral. For the counterclockwise prime spiral with startvalue 0 the general families of functions $f_{a,b,c}(n) = an^2 + bn + c$ thus become $f_{1,b,c}(n) = 1n^2 + bn + c$. When using the compass rose the value of b is synonymous to the winddirection.
Almost each function $f_{a,b,c}(n) = an^2 + bn + c$ appears as a horizontal, vertical or diagonal line only at a higher n.
For instance the sequence {41, 53, 67, 83, 101, ...} of the function $f_{1,11}(n_{SE}) = 1n^2 - 1n + 11$ (also known as $f_{1,-1,11}(n) = 1n^2 - 1n + 11$) starts at $n = 6$ and thus as part of the function $f(n) = n^2 + 11n + 41$. The function appears to be rich with prime numbers.

The function $f_{1,c}(n_{NE}) = 1n^2 + 1n + c$ with $c \in Z_0^-$ is separted from the function $f_{1,c}(n_{SE}) = 1n^2 - 1n + c$ with $c \in Z^+$.
When the graph is folded into a cone, with a shifted overlapping edge, it becomes a continuous spiral.

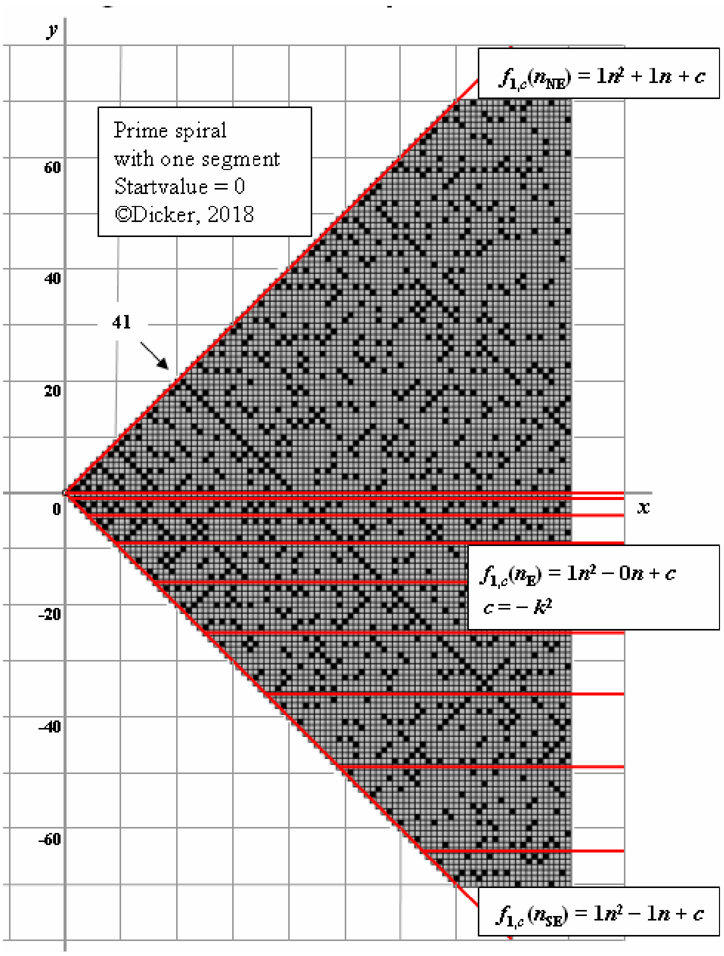

Fig. 6.2: Special factorable functions in the prime spiral with one segment.

Fig. 6.2 shows the special factorable functions that contain no prime numbers $> p_1$.
For instance the function $f_{1,-1}(n_E) = 1n^2 + 0n - 1$ with $c = -k^2 = -1$, can be written as $f_{1,-1}(n_E) = (n - 1)(n + 1)$.

The function $f_{1,41}(n_{SE}) = 1n^2 - 1n + 41$ is also known as Euler's famous formula for prime numbers $n^2 - n + 41$.
When this prime rich function intersects with a special factorable function, the natural number on that intersection is a composite number, see fig. 6.2.

A function value $f(n)$ is composite if $f(n) = d_A \bullet d_B$ with $d_x \mid f(n)$, $gcd(f(n), d_x) > 1 \;\; \forall d_x \in \{d_A, d_B\}$
If d_A is a divisor, then $d_A \mid f(n + d_A \bullet k)$ and $d_B(k) = f(n + d_A \bullet k) / d_A$ with $k \in \mathbf{N}_0$. Also if d_B is a divisor, then $d_B \mid f(n + d_B \bullet k)$ and $d_A(k) = f(n + d_B \bullet k) / d_B$. When $d_A(k) = d_B(k)$ the divisors generate new composite function values via a single pattern instead of a double pattern, see point a in the table below.

#	$f_{1,-1,41}(n) = 1n^2 - 1n + 41$	Coordinate	Pattern divisor A	Pattern divisor B
a	$f_{1,-1,41}(41) = 1681 = 41 \bullet 41$	A (41, 0)	$n = 41 + 41 \bullet k$	$n = 41 + 41 \bullet k$
b	$f_{1,-1,41}(42) = 1763 = 41 \bullet 43$	B (42, −1)	$n = 42 + 41 \bullet k$	$n = 42 + 43 \bullet k$
c	$f_{1,-1,41}(45) = 2021 = 43 \bullet 47$	C (45, −4)	$n = 45 + 43 \bullet k$	$n = 45 + 47 \bullet k$
d	$f_{1,-1,41}(50) = 2491 = 47 \bullet 53$	D (50, −9)	$n = 50 + 47 \bullet k$	$n = 50 + 53 \bullet k$
e	$f_{1,-1,41}(57) = 3233 = 53 \bullet 61$	E (57, −16)	$n = 57 + 53 \bullet k$	$n = 57 + 61 \bullet k$

7. Prime spirals with more segments.

The segmented prime spiral consists of segments of sequential natural numbers, who together with the other segments form a continuous spiral of natural numbers. The counterclockwise Ulam spiral with startvalue 0, named after Stanislaw Ulam in 1963, is a special seqential prime spiral with four segments. The Ulam spiral is a four-quarter spiral, and thus a prime spiral with four segments. Prime spirals with one, two or three segments can be visualized als partial Ulam spirals. In the segmented prime spirals the prime numbers have the tendency to line up along specific odd diagonals, while other odd diagonals hardly contain any prime numbers.

There are infinitely many segmented prime spirals. Fig. 2.1. shows an example of a counterclockwise prime spiral with five segments, while fig. 7.1. is a visualisation of a prime spiral with six segments.

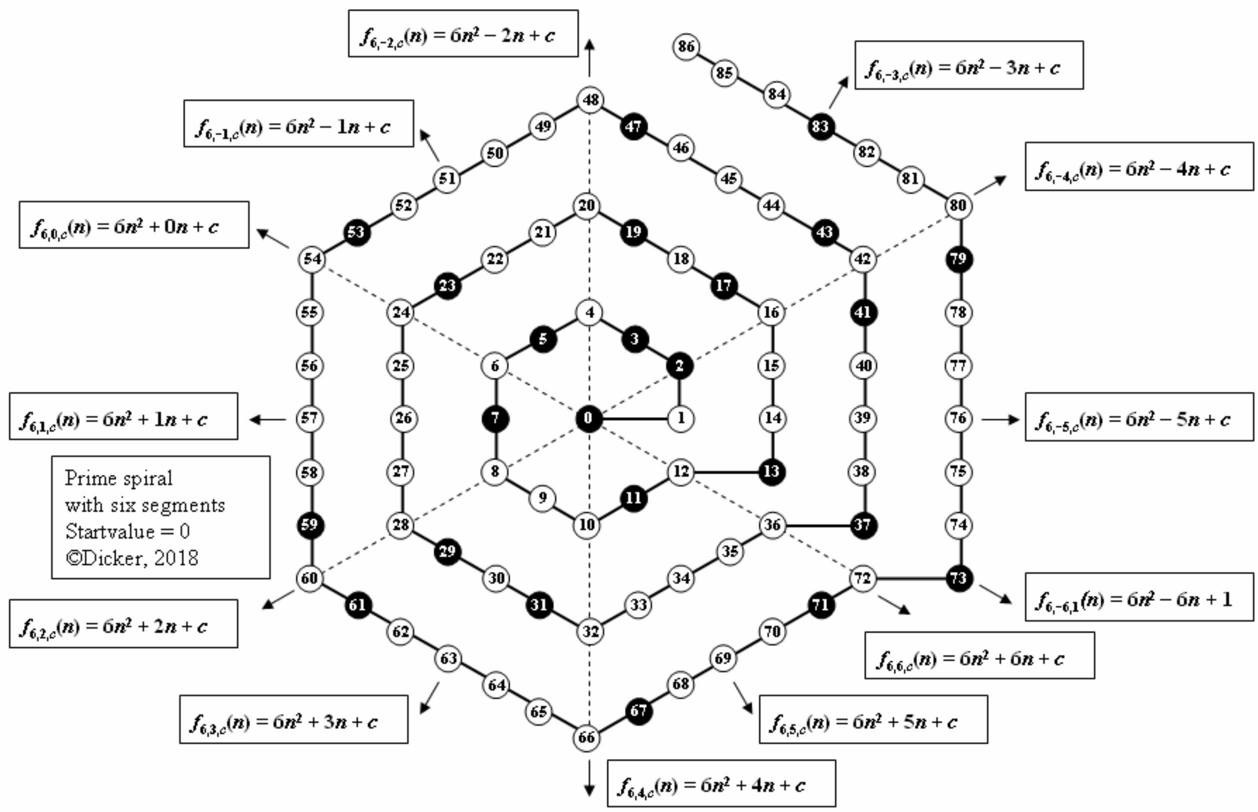

Fig. 7.1: A counterclockwise prime spiral with startvalue 0 and six segments.

The prime spiral with six segments (fig. 7.1) becomes a continuous spiral (see fig. 7.2) when the prime spiral is presented as a hexagram. There is now a seemingly seamless transition from the function $f_{6,6,c}(n) = 6n^2 + 6n + c$ with $c \in Z_0^-$ into the function $f_{6,-6,c}(n) = 6n^2 - 6n + c$ with $c \in Z^+$.

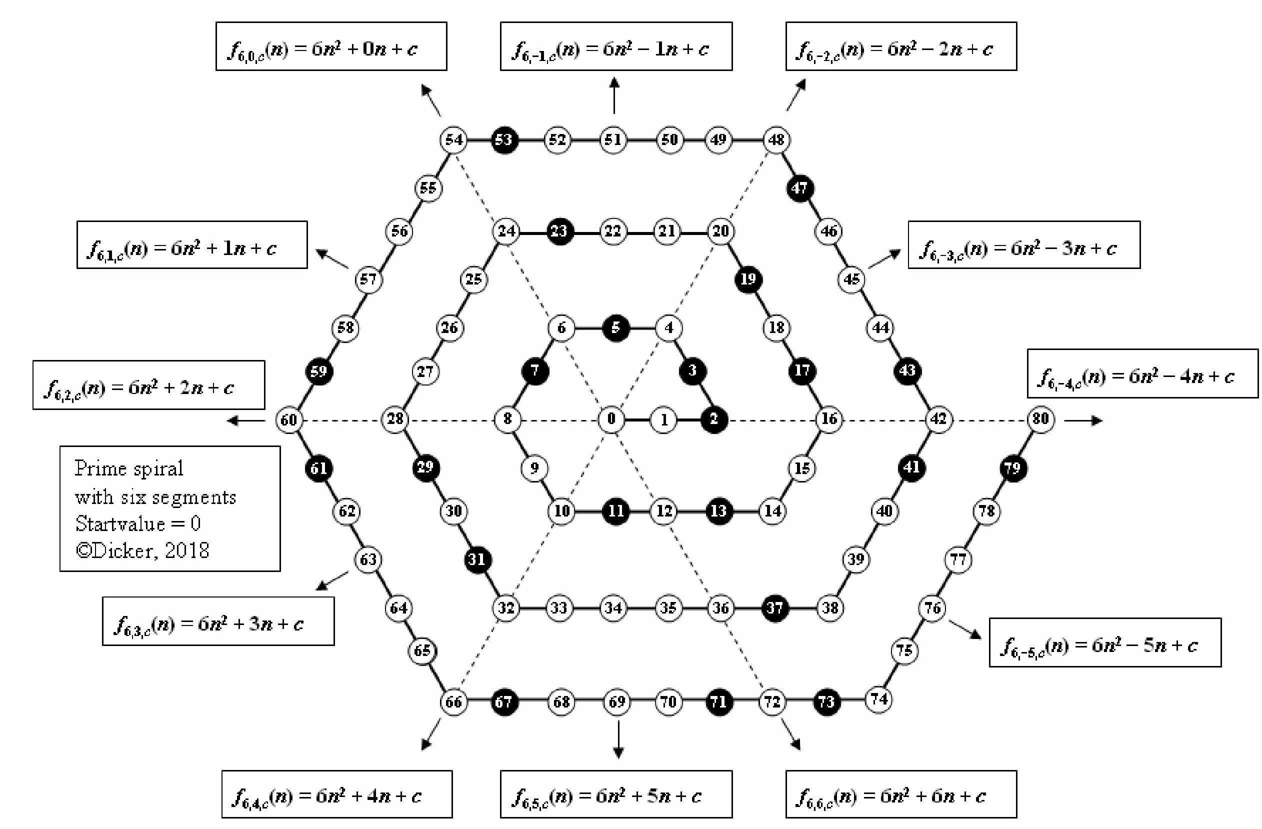

Fig. 7.2: A hexagram representation of a contiuous prime spiral with six segments.

8. Practical use of the segmented prime spirals.

The counterclockwise prime spiral with startvalue 0 and m segments is fully defined by the $(2m + 1)$ families of quadratic functions $f_{a,b,c}(n) = an^2 + bn + c$, with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$, $a = m$, $-a \le b \le a$ with $b \in Z$, and

$$\begin{cases} c \in Z_0^- & \text{if } b = a \\ c \in Z & \text{if } -a < b < a \\ c \in Z^+ & \text{if } b = -a \end{cases}$$

For $b = a$ the function $f_{a,b,c}(n) = an^2 + bn + c$ becomes $f_{a,b,c}(n) = an^2 + an + c$.
The translation $n \longmapsto n - 1$ then gives the function $f_{a,b,c}(n) = an^2 - an + c$ and thus $f_{a,b,c}(n) = an^2 - bn + c$.
The functions $f_{a,b,c}(n) = an^2 + bn + c$ and $f_{a,b,c}(n) = an^2 - bn + c$ give equal results, but for the translation $n \longmapsto n - 1$.

The definition of the families of functions within each of the infinitely many segmented prime spirals, makes it possible to identify specific quadratic functions that have a high ratio of prime numbers.

Euler's famous formula $n^2 - n + 41$ corresponds with the function $f_{1,-1,41}(n_{\mathrm{SE}}) = 1n^2 - 1n + 41$. Up to $f(n) = 10$^9 this function has a prime number ratio of 0.3590, see the table below. The function shows up as $f(n) = n^2 + 41n + 461$ in the Ulam one-quarter spiral as depicted in fig. 6.2, due to the translation $n \longmapsto n + 21$.
So far, for $f_{a,b,c}(n)$ up to 10^9 and $|c| < 400$, the highest prime number ratio is found for the function $f_{2,0,-199}(n_{\mathrm{NE}})$ which appears as $f(n) = 2n^2 + 400n + 19801$ in the Ulam two-quarter spiral.

a	b	c	total	primes	ratio
1	−1	41	31 624	11 356	0.3590...
		17	31 624	7 178	0.2269...
2	0	−199	22 361	8 868	0.3965...
		29	22 361	7 190	0.3215...
2	−2	127	22 361	7 397	0.3307...
3	3	−199	18 257	6 741	0.3692...
4	4	−397	15 811	6 242	0.3947...
	2	41	15 812	5 726	0.3621...
5	5	−167	14 142	4 522	0.3197...
5	−5	79	14 143	4 638	0.3279...
6	6	31	12 911	4 862	0.3766...

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