Analyzing segmented prime spirals.

1. Abstract.

The Ulam spiral is a prime spiral with four segments. The prime spiral with just one segment is analyzed to better understand the difference in density of prime numbers along horizontal, vertical and diagonal lines in the Ulam spiral

2. Families of functions in the segmented prime spirals.

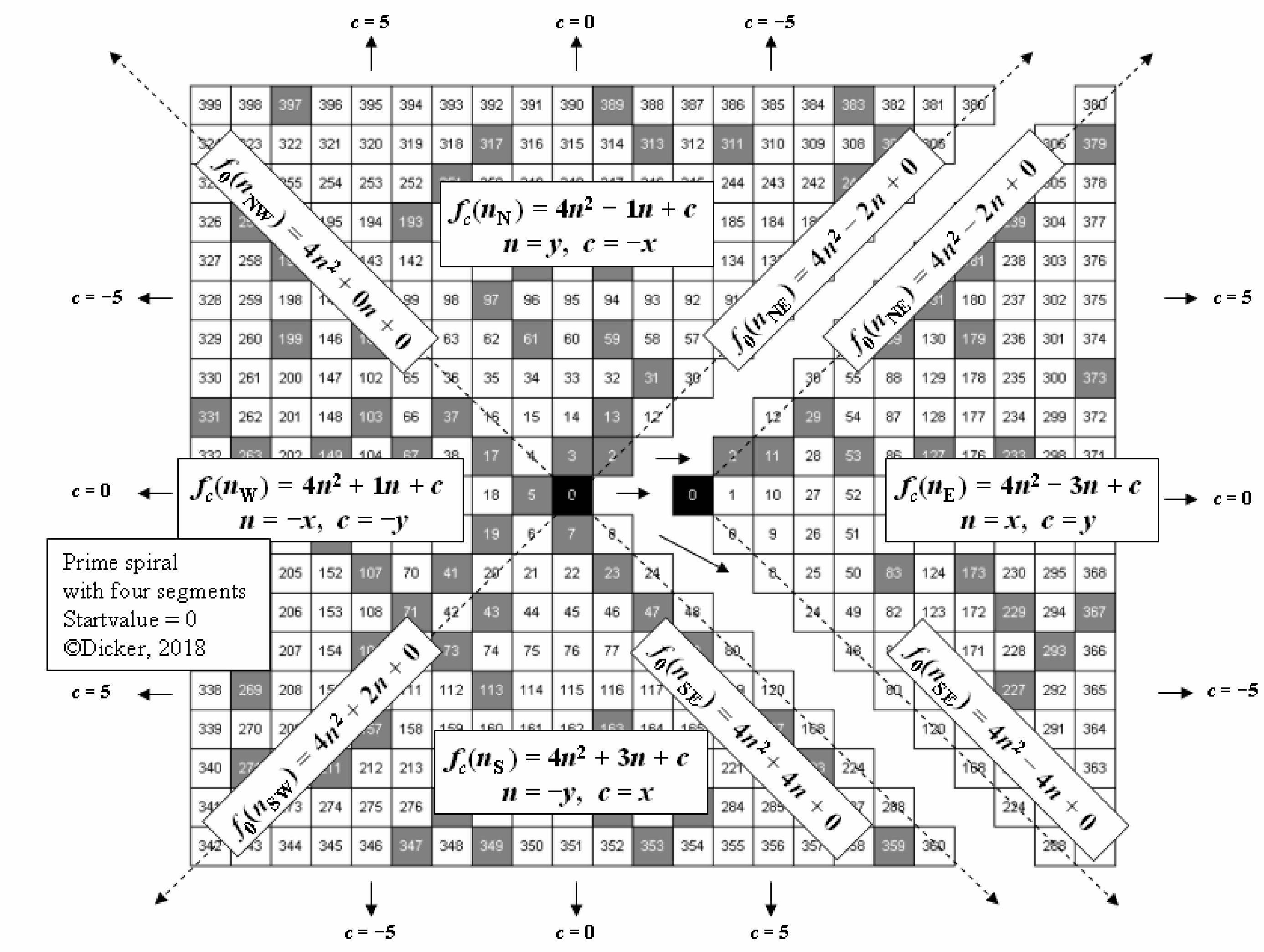

Fig. 1: The four segments in the counterclockwise Ulam spiral.

Counterclockwise prime spirals with startvalue 0 and m segments are fully defined by the $(2m + 1)$ families of quadratic functions $f_{a,b,c}(n) = an^2 + bn + c$, with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$, $a = m$, $-a \le b \le a$ with $b \in Z$, and

$$\begin{cases} c \in Z_0^- & \text{if } b = a \\ c \in Z & \text{if } -a < b < a \\ c \in Z^+ & \text{if } b = -a \end{cases}$$

3. Prime spiral with one segment.

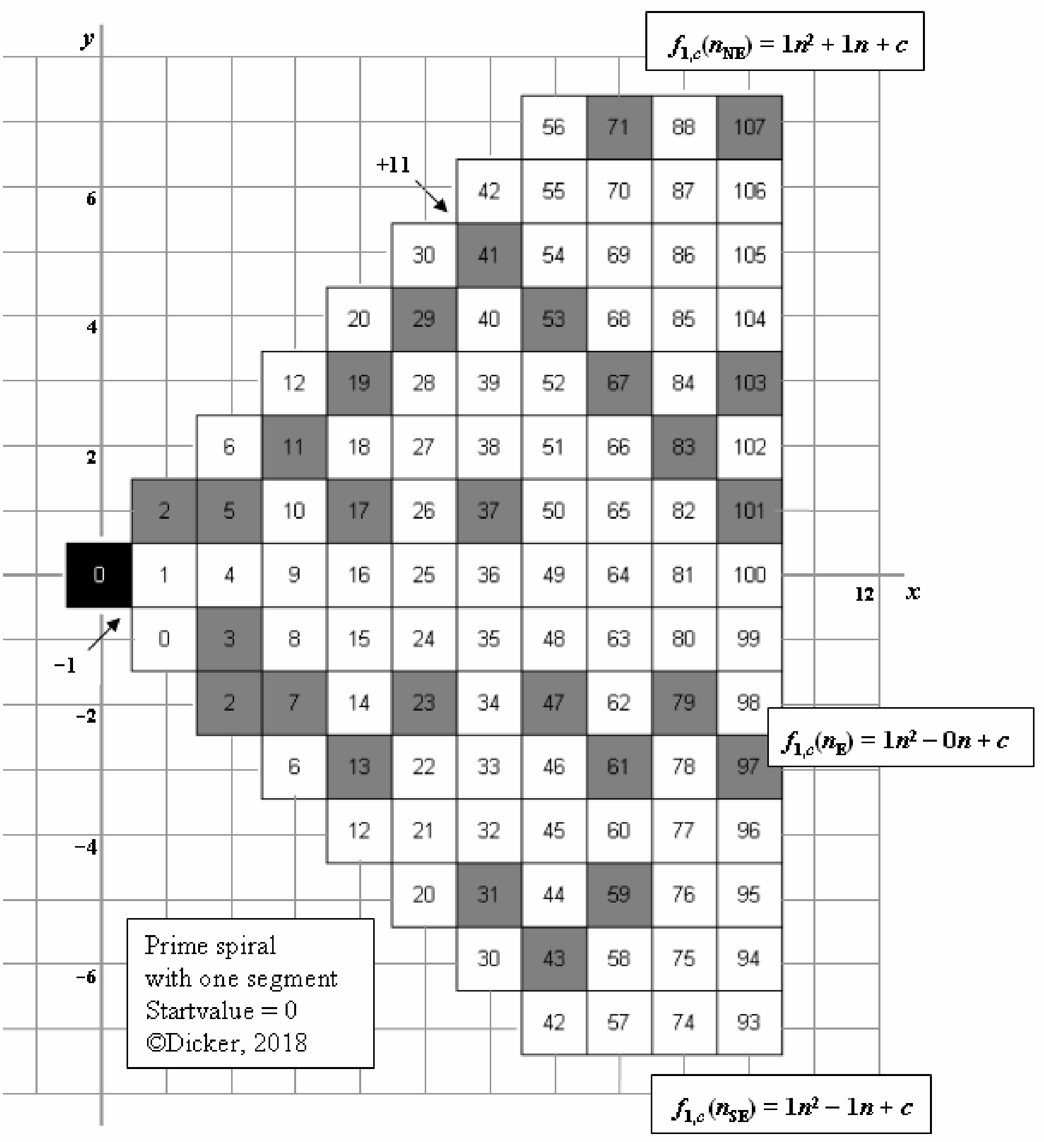

Fig. 2: The counterclockwise prime spiral with one segment.

For the prime spiral with one segment the families of functions are defined by $f_{1,b,c}(n) = 1n^2 + bn + c$ with $-a \le b \le a$.
When using the compass rose the value of b is synonymous to the winddirection.
The function $f_{1,-1,c}(n) = 1n^2 - 1n + c$ thus becomes $f_{1,c}(n_{SE}) = 1n^2 - 1n + c$.

Fig. 2 shows part of the infinitely large counterclockwise prime spiral with one segment. Based one the *E*-sector of the counterclockwise Ulam spiral (fig. 1) the natural numbers on the seams are duplicated.
So, elements of $f_{1,0}(n_{NE}) = 1n^2 + 1n + 0$ also appear as elements of $f_{1,0}(n_{SE}) = 1n^2 - 1n + 0$.
Note that the latter function does not comply with the definition of the families of functions, since $c \notin Z^+$ (see above).

Almost each function $f_{1,b,c}(n) = 1n^2 + bn + c$ appears as a horizontal, vertical or diagonal line only at a higher n.
For instance the sequence {41, 53, 67, 83, 101, ...} of the function $f_{1,11}(n_{SE}) = n^2 - n + 11$ starts at $n = 6$ and thus as part of the function $f(n) = n^2 + 11n + 41$. The function appears to be rich with prime numbers.
This function is not related to $f_{1,41}(n_{SE}) = n^2 - n + 41$, Euler's most famous prime number generator.

3. The role of prime number divisors.

A function value $f(n)$ is composite if $f(n) = d_A \bullet d_B$ with $d_x \mid f(n)$, $gcd(f(n), d_x) > 1 \ \forall d_x \in \{d_A, d_B\}$
If d_A is a divisor, then $d_A \mid f(n + d_A \bullet k)$ and $d_B(m) = f(n + d_A \bullet k) / d_A$ with $k \in \mathbf{N}_0$.
Also if d_B is a divisor, then $d_B \mid f(n + d_B \bullet k)$ and $d_A(k) = f(n + d_B \bullet k) / d_B$.

Given:

$d_A \mid f_{1,b,c}(n) = 1n^2 + bn + c$

For $n \longmapsto n + k$ with $k \in \mathbf{N}_0$ the families of functions

$f_{1,b,c}(n) = 1n^2 + bn + c$ become

$f_{1,b,c}(n) = 1n^2 + bn + c + k^2 + k(2n + b)$

Thus $d_A \mid n^2 + bn + c + k^2 + k(2n + b)$ when $d_A \mid k^2 + k(2n + b)$

And for $c \longmapsto c + m$ with $m \in \mathbf{Z}$ the families of functions

$f_{1,b,c}(n) = 1n^2 + bn + c$ become

$f_{1,b,c}(n) = 1n^2 + bn + c + m$

Thus $d_A \mid 1n^2 + bn + c + m$ when $d_A \mid m$

The divisor $d_A = 2$ eliminates all even natural numbers as possible prime number. Only odd natural numbers could be prime numbers, but for $p_1 = 2$. See fig. 3 below.

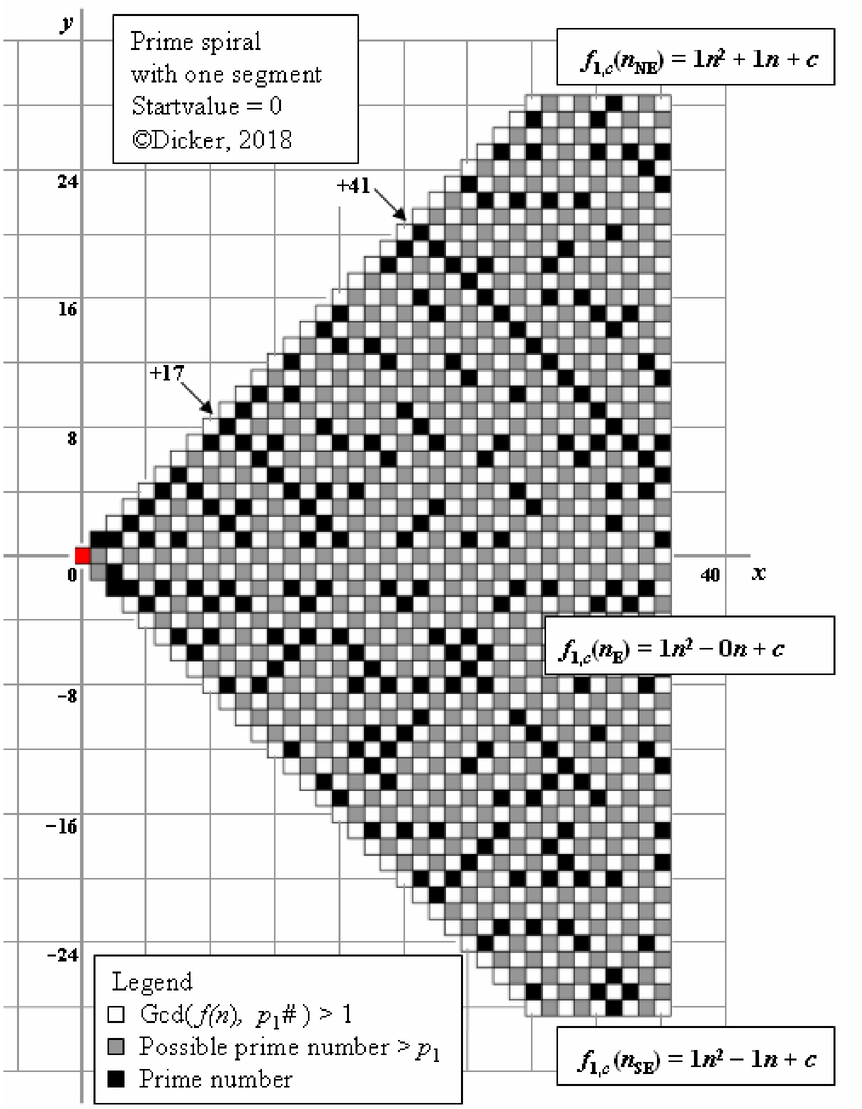

Fig. 3: Prime spiral with one segment: eliminating multiples of $p_1 = 2$.

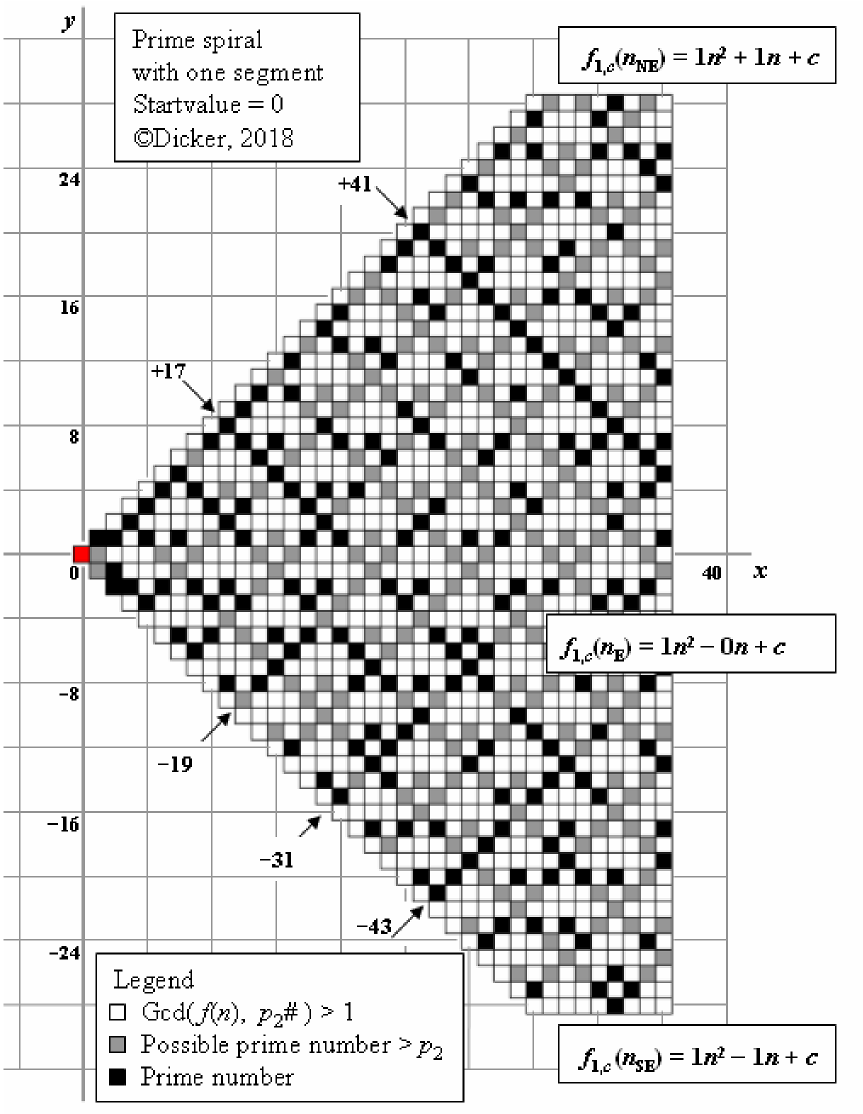

Fig. 4: Prime spiral with one segment: eliminating multiples of $p_1 = 2$ through $p_2 = 3$.

The divisor $d_A = 3$ eliminates all natural numbers with $\gcd(f(n), 3) > 1$ as possible prime number, but for $p_2 = 3$.
Given:

$d_A = 3$ and $d_A \mid f_{1,c}(n_{SE})$ with $f_{1,c}(n_{SE}) = 1n^2 - 1n + c$ and $c > 0$ ($c = 0$ is part of $f_{1,c}(n_{NE}) = 1n^2 + 1n + c$)

Then for $k \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$:

$d_A \mid f_{1,c}(n_{SE})$ for $n_{SE} = (1 + d_A \bullet k)$ and $c = 3 + d_A \bullet m$ or
$d_A \mid f_{1,c}(n_{SE})$ for $n_{SE} = (2 + d_A \bullet k)$ and $c = 3 + d_A \bullet m$ ($c = 3 + d_A \bullet 2m$ will suffice, due to odd/even numbers).

Thus:

when $3 \mid f_{1,c}(n_{SE})$ only one-third of the natural numbers could be prime numbers.

Also, given:

$d_A = 3$ and $d_A \mid f_{1,c}(n_{NE})$ with $f_{1,c}(n_{NE}) = 1n^2 + 1n + c$ and $c \leq 0$

Then for $k \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$:

$d_A \mid f_{1,c}(n_{NE})$ for $n_{NE} = (2 + d_A \bullet k)$ and $c = -3 - d_A \bullet m$ or
$d_A \mid f_{1,c}(n_{NE})$ for $n_{NE} = (3 + d_A \bullet k)$ and $c = -3 - d_A \bullet m$ ($c = -3 - d_A \bullet 2m$ will suffice, due to odd/even numbers).

Thus:

when $3 \mid f_{1,c}(n_{NE})$ only one-third of the natural numbers could be prime numbers.

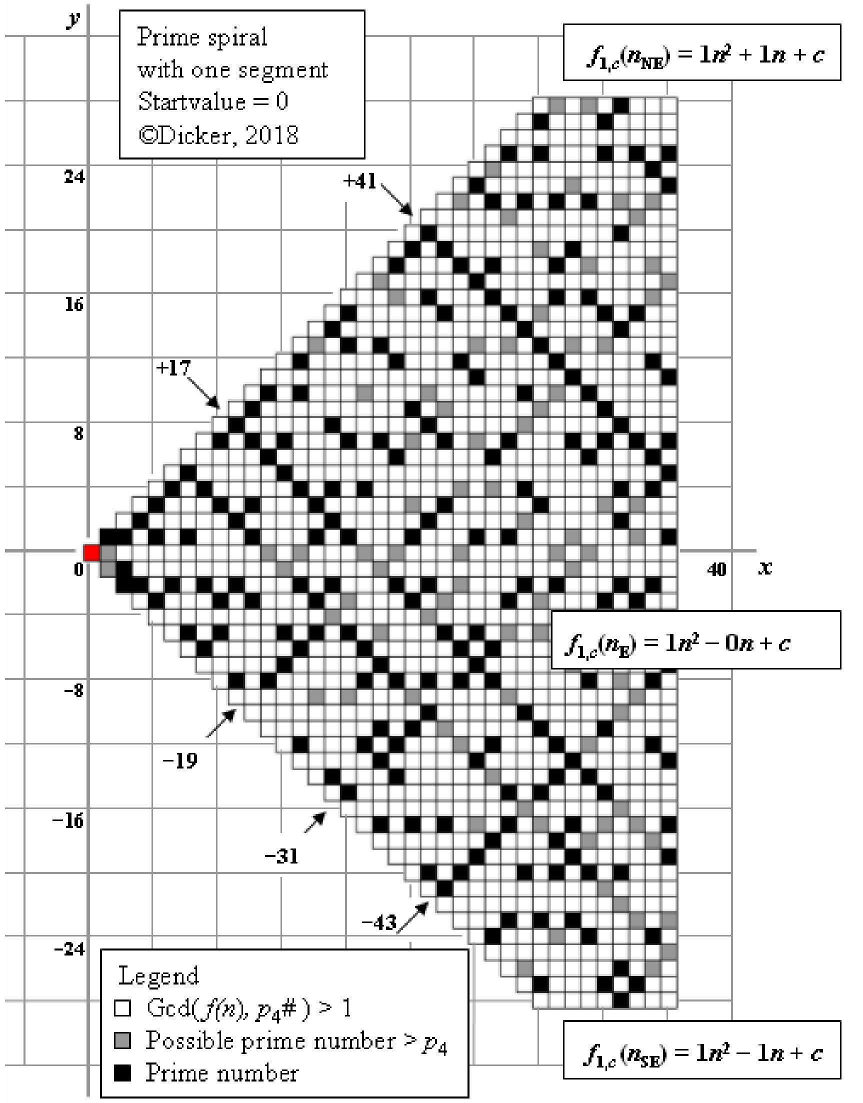

Fig. 5: Prime spiral with one segment: eliminating multiples of $p_1 = 2$ through $p_4 = 7$.

Fig. 5 shows distinct patterns of (possible) prime numbers when all natural numbers with $\gcd(f(n), p_4\#) > 1$ are removed. The NE diagonals are further hindered by special factorable functions $f_{1,c}(n_E) = 1n^2 - 0n + c$ with $c = -k^2$ and $k \in \mathbf{N}$, like $f_{1,0}(n_E) = 1n^2 - 0n - 0$ and $f_{1,-1}(n_E) = 1n^2 - 0n - 1$.

Fig. 6a and 6b show the proportion of prime numbers up to $f(n) = 10^9$ on several NE and SE diagonals.

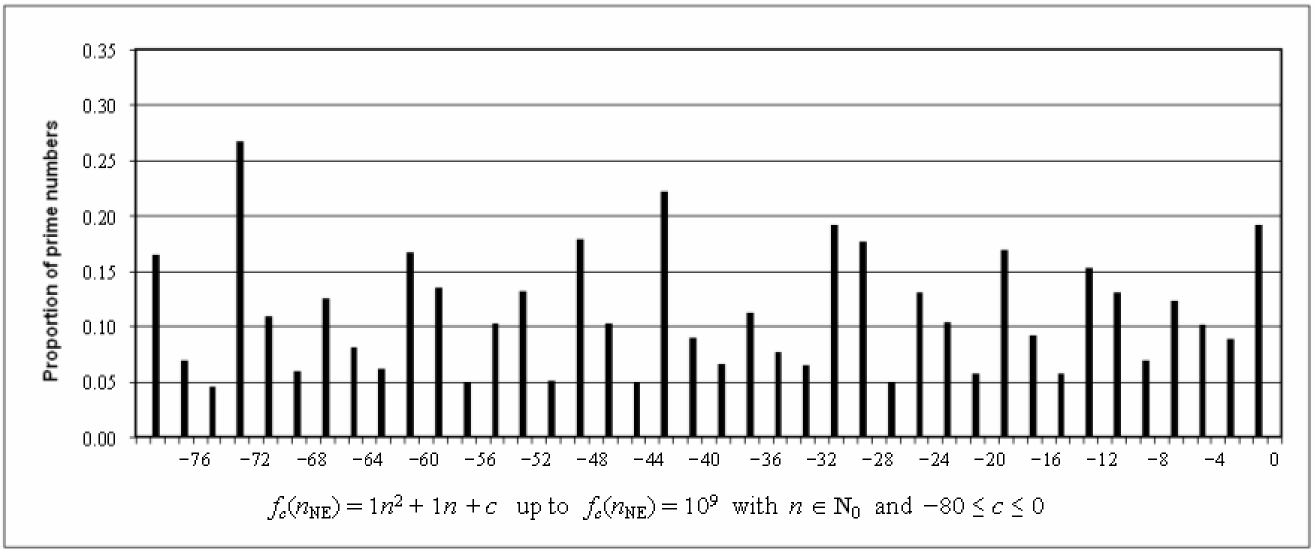

Fig. 6a: Prime number density on NE diagonals.

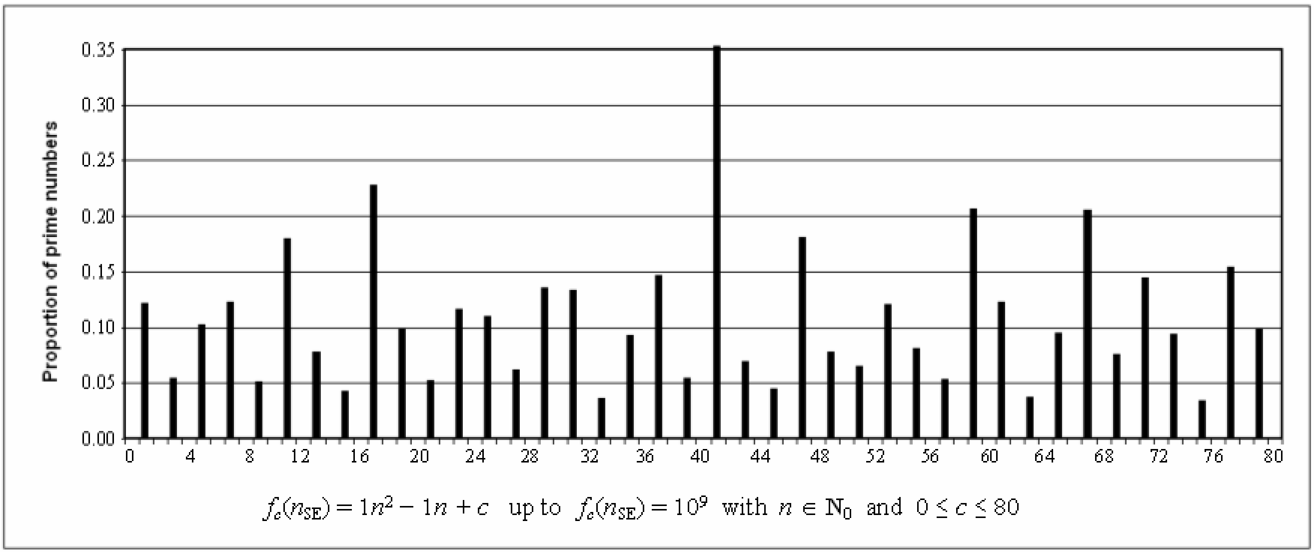

Fig. 6b: Prime number density on SE diagonals.